Cryptography

Lecture 4: Pseudorandom Generators

November 5, 2024

Overview

Contents

1. Classical cryptography
   (Shift & Vigenère cipher, one-time pad, perfect secrecy)
2. Security definitions & threat models
   (Computational security, CPA & CCA)
3. Private-key cryptography
   (Message authentication, hash functions, primitives, relevant ciphers)
4. Public-key cryptography
   (Assumptions, key management, digital signatures, relevant ciphers)

Practical Considerations

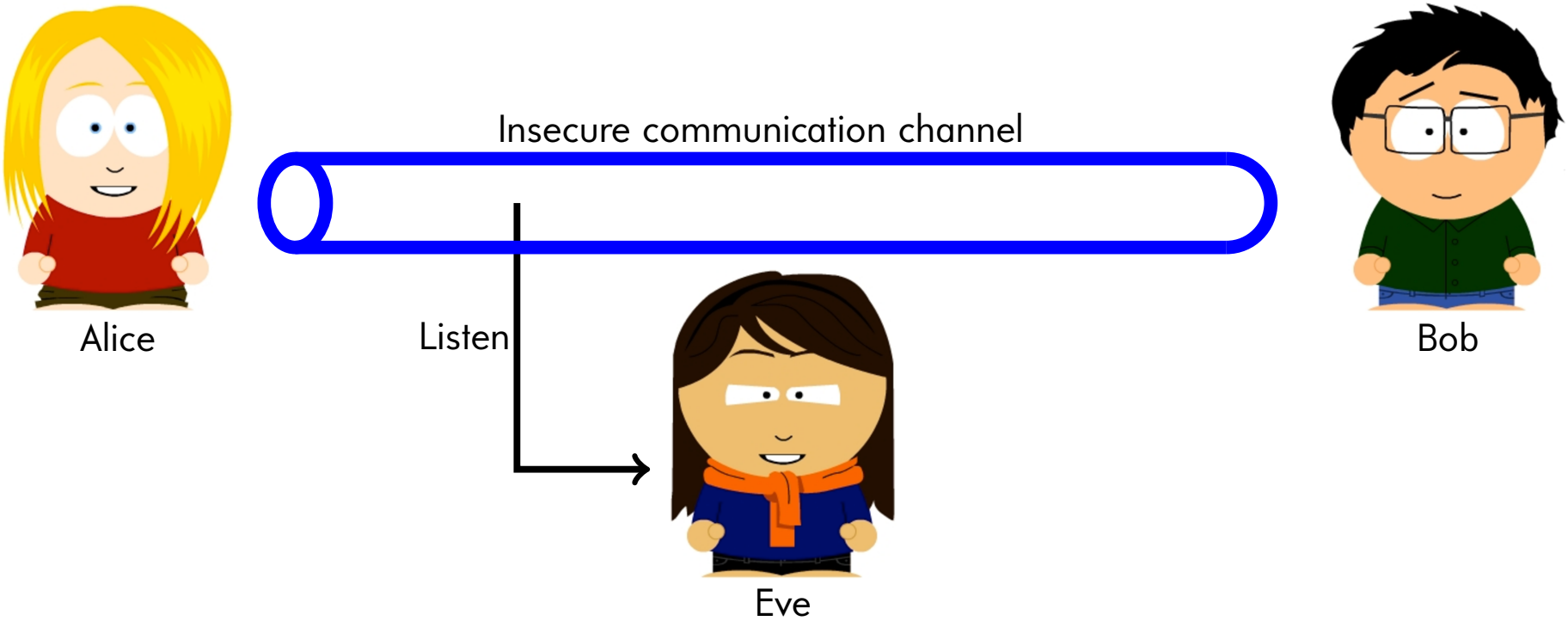
Perfect secrecy

- Unbreakable if used correctly
- Expensive due to huge key space & required key exchanges
- Only feasible for extremely secret information

Practical secrecy

- Allow key reuse
- Weaken requirements
  - ~~Perfect correctness~~
  - Threat model
  - Perfect secrecy
- Allow control over security

Threat Models



Threat model = What can Eve do?

Threat Models

Goal of attacker

Decrypt current ciphertext

Encryption threat models (in increasing power)

1. Ciphertext-only attack = Eve only collects ciphertexts but has no access to additional information (eavesdropper attack) (we essentially used this model so far)
2. Known-plaintext attack = Eve also knows some plaintext-ciphertext pairs for same key (informed guesses often possible)
3. Chosen-plaintext attack = Eve knows plaintext-ciphertext pairs for plaintexts of her choice (Alice encrypts messages of Eve's choice)
4. Chosen-ciphertext attack = Eve also knows plaintext-ciphertext pairs for ciphertexts of her choice (Bob decrypts ciphertexts of Eve's choice)

Threat Models

Computational capabilities of attacker

- Unrestricted (used so far) (anything goes)
- Computable (any algorithm)
- Efficient (i.e. polynomial-time) (any polynomial-time algorithm)

Notes

- Key space usually exponential size in key length (2^n keys of length n)
- Full brute-force attack takes exponential-time in key length
  (assuming efficient decryption & recognition of valid messages)
- Partial brute-force attack possible in polynomial-time
  (under same assumptions)

$\rightarrow$ Weakening of perfect secrecy also necessary

Efficient Encryption Scheme

§3.6 Definition (cf. §2.4; Efficient private-key encryption scheme)

(Efficient) private-key encryption scheme is triple $(\mathsf{gen}, \mathsf{enc}, \mathsf{dec})$ of probabilistic polynomial-time computable functions

- $\mathsf{gen}\colon R \times \{1\}^* \to K$ i.e. family $(\mathrm{P}_K^n)_{n\in\mathbb{N}}$ of distributions $\mathrm{P}_K^n\colon K \to [0,1]$
- $\mathsf{enc}\colon R \times K \times M \to C$ and $\mathsf{dec}\colon K \times C \to (M \cup \{\bot\})$
- $|\mathsf{gen}(r, 1^n)| \geq n$ for all $r \in R$ and $n \in \mathbb{N}$
- $\mathsf{dec}_k(c) = m$ for all $k \in \mathrm{ran}(\mathsf{gen})$, $m \in M$, and $c \in C$ with $\mathrm{P}\big[c \leftarrow \mathsf{enc}_k(m)\big] \neq 0$ (perfect correctness)

Notes

- Security parameter n = additional parameter of gen
- Generated keys have at least length n
- Decrypt failure indicated by $\bot$

Efficient Encryption Scheme

Example: Vigenère cipher

- $\mathsf{P}_K^n(k) = 2^{-n}$ for all $k \in \{0,1\}^n$ and $\mathsf{P}_K^n(k) = 0$ otherwise
  (uniform random selection of key of length n)
- $\mathsf{enc}_k(m_1, \ldots, m_\ell) = (m_1 \oplus k_{0 \bmod n}, \ldots, m_\ell \oplus k_{(\ell-1) \bmod n})$
  with key $k = (k_0, \ldots, k_{n-1})$
- $\mathsf{dec}_k = \mathsf{enc}_k$
- These functions are certainly polynomial-time computable

Note

- Shift cipher offers no obvious security parameter

Efficient Encryption Scheme

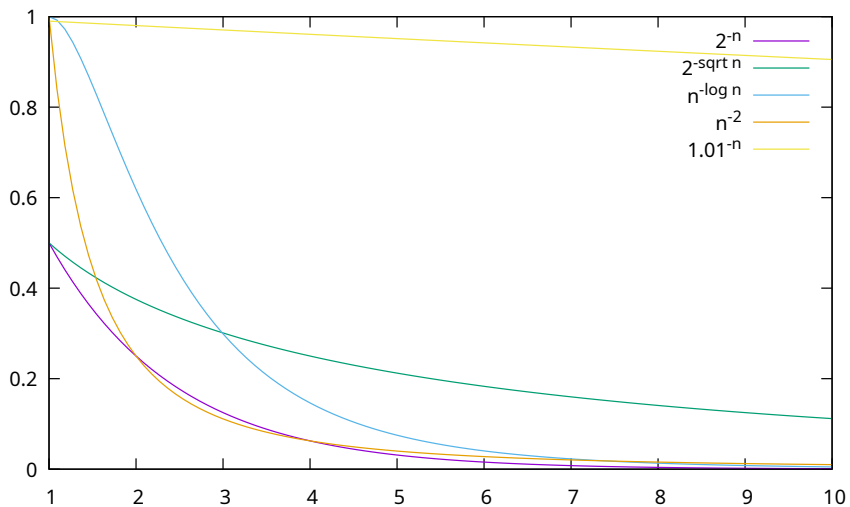
Notes

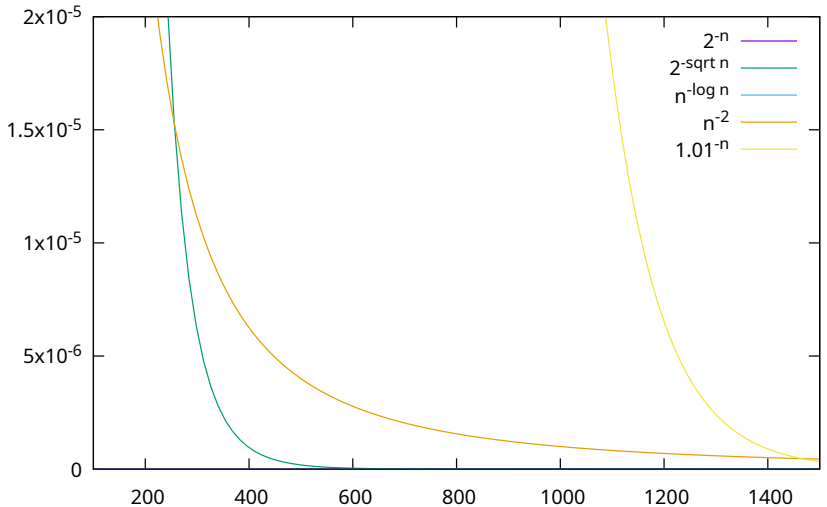
- Asymptotic approach (polynomial time $\neq$ quick)
  (polynomial time
  = eventually small complexity increase for small input size increase)
- PPT = probabilistic polynomial-time (computable)
- Also allow eventually small advantage of attacker
  $\rightarrow$ negligible functions

Computational Indistinguishability

§3.7 Definition (Negligible function)

Function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is negligible if for every positive polynomial p there exists $n_0 \in \mathbb{N}$ such that $f(n) < p(n)^{-1}$ for all $n \in \mathbb{N}$ with $n \geq n_0$





Notation

- We frequently use sequence notation for functions with domain $\mathbb{N}$ i.e. $(2^{-x})_{x \in \mathbb{N}}$ for function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ with $f(x) = 2^{-x}$

Efficient Encryption Scheme

Questions

- Which of those functions are negligible?
  - $(2^{-n})_{n\in\mathbb{N}}$
  - $(2^{-\sqrt{n}})_{n\in\mathbb{N}}$
  - $(n^{-\log n})_{n\in\mathbb{N}}$
  - $(n^{-2})_{n\in\mathbb{N}}$
  - $(1.01^{-n})_{n\in\mathbb{N}}$
- Is the sum/product of 2 negligible functions negligible?
- Is the sum/product of negligible function with any function negligible?

Computational Indistinguishability

§4.1 Definition (Almost bound & almost equality)

Function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is (asymptotically) almost bounded by $b \in \mathbb{R}_{\geq 0}$, written $f \preccurlyeq b$, if there exists negligible function $\tilde{0} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(n) \leq b + \tilde{0}(n)$ for all $n \in \mathbb{N}$.

Functions $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ are (asymptotically) almost equal, written $f \simeq g$, if

$$\big(|f(n) - g(n)|\big)_{n \in \mathbb{N}} \preccurlyeq 0$$

Notes

- We generally use $\tilde{0}$ for negligible functions
- $h \preccurlyeq 0$ iff h is negligible
- $f \simeq g$ iff $|f - g|$ is negligible (absolute value of pointwise difference)

Computational Indistinguishability

§4.2 Definition (Indistinguishability game)

Let $n \in \mathbb{N}$, $\mathcal{E} = (\mathrm{gen}, \mathrm{enc}, \mathrm{dec})$ efficient private-key encryption scheme and $\mathcal{A}$ stateful algorithm. Adversarial indistinguishability game $\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)$ is

1. Security parameter 1^n sent to adversary $\mathcal{A}$
2. $\mathcal{A}$ selects messages $m_0, m_1 \in M$ with $|m_0| = |m_1|$
3. $k \leftarrow \mathrm{gen}(1^n)$, uniform random bit $b \in \{0,1\}$
   $c \leftarrow \mathrm{enc}_k(m_b)$ sent to $\mathcal{A}$ (challenge ciphertext)
4. $\mathcal{A}$ outputs bit $b' \in \{0,1\}$
5. Return 1 if $b = b'$ ($\mathcal{A}$ wins)
   Return 0 otherwise

Changes

- $\mathcal{A}$ receives security parameter n & selects messages of equal length
- Key generation with security parameter

Computational Indistinguishability

Alice

Eve

Computational Indistinguishability

Alice

Security parameter 1^n

Eve

Computational Indistinguishability

Alice

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Eve

Computational Indistinguishability

Alice
$k \leftarrow \text{gen}(1^n)$
random b

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Eve

Computational Indistinguishability

Alice

$k \leftarrow \text{gen}(1^n)$

random b

$c \leftarrow \text{enc}_k(m_b)$

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Eve

Computational Indistinguishability

Alice

$k \leftarrow \text{gen}(1^n)$

random b

$c \leftarrow \text{enc}_k(m_b)$

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Ciphertext c

Eve

Computational Indistinguishability

Alice
$k \leftarrow \text{gen}(1^n)$
random b
$c \leftarrow \text{enc}_k(m_b)$

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Ciphertext c

Eve
Determines b'

Computational Indistinguishability

Alice
$k \leftarrow \text{gen}(1^n)$
random b
$c \leftarrow \text{enc}_k(m_b)$

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Ciphertext c

Bit b'

Eve
Determines b'

Computational Indistinguishability

Alice

$k \leftarrow \text{gen}(1^n)$

random b

$c \leftarrow \text{enc}_k(m_b)$

Return $b \stackrel{?}{=} b'$

Security parameter 1^n

Messages m_0, m_1 with $|m_0| = |m_1|$

Ciphertext c

Bit b'

Eve

Determines b'

Computational Indistinguishability

§4.3 Definition (EAV-secure)

Private-key encryption scheme $\mathcal{E} = (\mathsf{gen}, \mathsf{enc}, \mathsf{dec})$ is EAV-secure if for every PPT algorithm $\mathcal{A}$ (in security parameter)

$$\left(\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)\right]\right)_{n\in\mathbb{N}} \preccurlyeq \frac{1}{2}$$

Notes

- $\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)\right]$ = probability that $\mathcal{A}$ wins for security parameter n
- EAV-secure = no PPT algorithm $\mathcal{A}$ can do much better asymptotically
- Can replace $\left(\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)\right]\right)_{n\in\mathbb{N}} \preccurlyeq \frac{1}{2}$ by $\left(\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)\right]\right)_{n\in\mathbb{N}} \simeq \frac{1}{2}$?
- Careful: no concrete guarantees

Computational Indistinguishability

Example: Vigenère cipher $\mathcal{E}$

- Adversary $\mathcal{A}$ receives 1^n and
  selects messages $m_0 = a^n a^n$ and $m_1 = a^n b^n$ of equal length
- Adversary $\mathcal{A}$ receives ciphertext $c = c_1 c_2$ with $|c_1| = n = |c_2|$
  - Returns 0 if $c_1 = c_2$
  - Returns 1 otherwise
- Adversary $\mathcal{A}$ obviously PPT

$$
\begin{aligned}
&\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n)\right] \\
&= \frac{1}{2}\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n) \mid b = 0\right] + \frac{1}{2}\mathrm{P}\left[\mathrm{PrivK}^{\mathrm{one}}_{\mathcal{E},\mathcal{A}}(n) \mid b = 1\right] \\
&= \frac{1}{2}\mathrm{P}[c_1 = c_2 \mid b = 0] + \frac{1}{2}\mathrm{P}[c_1 \neq c_2 \mid b = 1] \\
&= \frac{1}{2} + \frac{1}{2} = 1 \not\leq \frac{1}{2}
\end{aligned}
$$

$\rightarrow$ Vigenère cipher not EAV-secure

Randomness

Question: Which is uniformly random 32 bit string?

- 0101 0101 0101 0101 0101 0101 0101 0101
- 1010 0101 1010 1001 1110 1011 0110 1101
- 0000 0000 0000 0000 0000 0000 0000 0000

Answer

- Each probability 2^{-32} if generated uniformly
- Uniform randomness is property of distribution (not drawn strings)
- Uniform distribution U_n with $U_n(b) = 2^{-n}$ for all $b \in \{0,1\}^n$

Randomness

Remarks

- We accept casual terminology
  uniformly random $b = b$ drawn from uniformly random distribution
- Given distribution D we write $d \leftarrow D$ for d sampled from D
- Instead of "uniform random bit $b \in \{0,1\}$" we write $b \leftarrow U_1$
- We agree on universal security parameter $n \in \mathbb{N}$
  and omit $(\cdot)_{n\in\mathbb{N}}$ whenever context demands sequence

$$\mathsf{P}\left[\mathsf{PrivK}^{\mathsf{one}}_{\mathcal{E},\mathcal{A}}(n)\right] \preccurlyeq \frac{1}{2} \quad \text{instead of} \quad \left(\mathsf{P}\left[\mathsf{PrivK}^{\mathsf{one}}_{\mathcal{E},\mathcal{A}}(n)\right]\right)_{n\in\mathbb{N}} \preccurlyeq \frac{1}{2}$$

Pseudorandom Generator

§4.4 Definition (Pseudorandom generator)

Let p be polynomial and G deterministic polynomial-time algorithm such that $G(s) \in \{0,1\}^{p(n)}$ for all $s \in \{0,1\}^n$ and $n \in \mathbb{N}$.
G is pseudorandom generator if

- $p(n) > n$ for all $n \geq 1$ (expansive)
- For any PPT algorithm $D\colon \{0,1\}^* \to \{0,1\}$

$$\sum_{s \in \{0,1\}^n} \frac{\mathrm{P}(D(G(s)))}{2^n} \simeq \sum_{r \in \{0,1\}^{p(n)}} \frac{\mathrm{P}(D(r))}{2^{p(n)}}$$

Notes

- G extends short random string s into long pseudorandom string $G(s)$
- Long pseudorandom string $G(s)$ should be computationally indistinguishable from long, truely random string r

Pseudorandom Generator

$$\sum_{s\in\{0,1\}^n} \frac{\mathrm{P}\big(D(G(s))\big)}{2^n} = \mathrm{E}\Big[P\Big(D\big(G(s)\big)\Big)\Big]_{s\leftarrow U_n}$$

$$\sum_{r\in\{0,1\}^{p(n)}} \frac{\mathrm{P}\big(D(r)\big)}{2^{p(n)}} = \mathrm{E}\Big[P\Big(D\big(r\big)\Big)\Big]_{r\leftarrow U_{p(n)}}$$

We compare success probabilities of these 2 algorithms

1. $s \leftarrow U_n$
2. $r' = G(s)$
3. $D(r')$

1. $r \leftarrow U_{p(n)}$
2. $D(r)$

Pseudorandom Generator

Example

- $G(b) = bb^R$ with polynomial $p(x) = 2x$
- Expansive & deterministically computable in polynomial time
- Stupid distinguisher D with input $d_1 \cdots d_{2n}$ for $d_1, \ldots, d_{2n} \in \{0,1\}$
  1. $i \leftarrow (n^{-1})_{a \in \{1,\ldots,n\}}$ (draw i from uniform distribution on $\{1, \ldots, n\}$)
  2. Return 1 if $d_i = d_{n+i}$; return 0 otherwise
- $r \leftarrow U_{2n}$; $D(r)$ succeeds with probability $\frac{1}{2}$ (compares two random bits)
- $s \leftarrow U_n$; $r' = G(s)$; $D(r')$ succeeds
  - with probability $\frac{1}{2}$ if n is even (compares two random bits)
  - with probability $\frac{1}{n} + \frac{(n-1)}{n} \cdot \frac{1}{2} = \frac{2+n-1}{2n} = \frac{n+1}{2n}$ if n is odd
- Hence $\frac{n+1}{2n} - \frac{1}{2} = \frac{n+1-n}{2n} = \frac{1}{2}n^{-1}$, which is not negligible

$\rightarrow$ G is not pseudorandom generator

Pseudorandom Generator

Example

- $G(b) = bb^R$ with polynomial $p(x) = 2x$
- Better distinguisher D' with input $d_1 \cdots d_{2n}$ for $d_1, \ldots, d_{2n} \in \{0,1\}$
  1. Return 1 if $d_1 = d_{2n}$
  2. Return 0 otherwise
- $r \leftarrow U_{2n}$; $D'(r)$ succeeds with probability $\frac{1}{2}$
- $s \leftarrow U_n$; $r' = G(s)$; $D'(r')$ always succeeds

$\rightarrow$ G is not pseudorandom generator

Pseudorandom Generator

Why "pseudorandom"

- Expansion is $p(x) \geq x + 1$ for pseudorandom generator G
- Each "random" bit sequence of length $p(n)$ has probability $2^{-p(n)}$
- $|\{G(s) \mid s \in \{0,1\}^n\}| \leq 2^n$,
  so at least $2^{p(n)} - 2^n \geq 2^n$ sequences impossible

- Universal distinguisher D with input $d \in \{0,1\}^{p(n)}$
  1. Return 1 if there exists $s \in \{0,1\}^n$ such that $G(s) = d$
  2. Return 0 otherwise

- $r \leftarrow U_{p(n)}$; $D(r)$ succeeds with probability at most $\frac{2^n}{2^{p(n)}} \leq \frac{1}{2}$
- $s \leftarrow U_n$; $r' = G(s)$; $D(r')$ always succeeds
- Difference is not negligible, but D is not known to be polynomial time
  (brute-force attack against pseudorandom generator)

Summary

- Threat models
- Negligible functions
- Computational indistinguishability & EAV-security
- Pseudorandom generators